Breaking a Matrix into Simple Pieces

A First Look at Matrix Factorizations

BAMS project - Algebra and Geometry

Abstract.

In class we learn that, when A is invertible, the solution of $Ax = b$ is $x = A^{-1}b$. Yet software that solves linear systems almost never computes A^{-1}. Instead, it breaks A into a product of simple matrices, such as triangular ones, and solves one easy piece at a time. **This project asks how this works, and why it is better.** One surprise along the way: a computer that rounds its numbers can get a 2×2 system completely wrong, and the way it breaks the matrix decides whether it does.

Along the way, factorizations also show what a matrix does geometrically. Multiply every point of the unit circle by an invertible 2×2 matrix: the points form an ellipse (Figure 1). Its axes lead to a surprising discovery: is every linear map of the plane just a rotation, a stretch and another rotation?

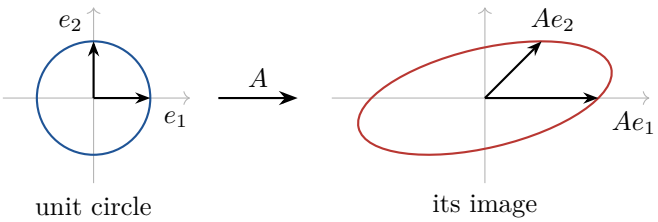


Figure 1. The matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ sends the unit circle to an ellipse. Where are its axes, and what do they tell us about A?

Prerequisites / Difficulty.

Accessible. First-year linear algebra is enough: Gaussian elimination, matrix products, inverses and transposes, the dot product and quadratic forms, plus maxima and minima from first-year calculus. Everything is computed by hand on 2×2 and 3×3 matrices. No numerical analysis and no programming is required.

Starting materials.

- Gilbert Strang, *Introduction to Linear Algebra*, 6th ed., Wellesley-Cambridge Press (2023): Sections 2.3–2.4 ($A = LU$), 4.4 (Gram–Schmidt and QR), 6.3 (positive definite matrices), Chapter 7 (singular value decomposition).
- Gilbert Strang, MIT 18.06 Linear Algebra, video lectures, in particular "Factorization into $A = LU$".
- Lloyd N. Trefethen and David Bau, *Numerical Linear Algebra*, SIAM (1997), Lectures 20, 21 and 23, for those who want to go further.

Possible directions.

The common path. The whole group works through it together. It answers the question of the project, and it is a complete project on its own.

- **Starting point: elimination is a factorization.** Write each step of Gaussian elimination as a matrix, and discover that elimination splits A into two triangular matrices, $A = LU$.

- **A. Why computers exchange rows.** Round every number to three digits, watch a small system go wrong, and find the rule that fixes it.
- **B. Completing the square.** Every symmetric matrix gives an expression in x and y: the matrix $\left(\begin{smallmatrix} 4 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ gives $4x^2 + 4xy + 3y^2$. Completing the square rewrites it as $(2x + y)^2 + 2y^2$, and this splits the matrix into two triangular pieces. Find out how, and when it works.
- **C. Rotate, then solve.** Make the columns of a matrix perpendicular, and discover that this splits it into a rotation and a triangular piece. Ask why this is safer than elimination.
- **D. What a matrix does to a circle.** Find the axes of the ellipse in Figure 1, and decide whether every linear map of the plane is a rotation, a stretch and another rotation.

Further directions (optional). Groups that finish early, or that want a different angle, can continue with one of these, or with an idea of their own.

- **Compressing a picture.** Build a small picture as the sum of two simple patterns, and see how the singular value decomposition separates them. Explain how the same idea compresses real images.
- **The best line through data.** Find the line that best fits three or four points that are not aligned, using QR.
- **How much can a matrix magnify errors?** Compare the longest and shortest axes of the ellipse, and use them to explain the system that went wrong in A.
- **Multiplying faster.** Multiply two 2×2 matrices with 7 multiplications instead of 8 (Strassen's trick), and find out what this saves for larger matrices.
- **Your own idea.** Any question that comes up along the way is welcome.

Expected outcome.

The group writes a short paper in LaTeX. The paper follows the common path, with one section for each step and examples computed by hand, and answers the question of the project with a comparison of the factorizations. If the group follows a further direction, a final section presents it. The members share the writing, each drafting one or two sections. After the groups are formed, they receive a detailed guide with step-by-step tasks for the common path.

Contacts.

For questions, ideas or further discussion, contact Luigi Lanni:
luigi.lanni@studbocconi.it
Phone / WhatsApp: +39 329 446 5773.