Optimal Growth:
Convergence, Stability and Cycles

BAMS project / course insight - Advanced Mathematics

Abstract.

In Advanced Mathematics, you have seen how dynamic programming characterizes optimal decisions through a Bellman equation. In growth economics, these decisions generate a path of capital and consumption. This project studies the long-run behavior of that path. What makes an economy converge to a steady state, how fast does it approach it, and can optimal decisions instead generate persistent cycles?

A starting point could be the one-sector growth model and its optimal capital policy. From there, the project develops the theory of one-dimensional discrete dynamical systems to investigate stability, convergence and periodic behavior. The comparison with a two-sector economy gives a reason to study these tools: optimality alone does not establish that an economy approaches a steady state. The economic question is which features of saving and production lead to different long-run outcomes.

Prerequisites / Difficulty.

- Sequences, continuity and fixed points from Advanced Mathematics.
- Dynamic programming and optimal policy functions from the course.
- Basic differentiation for the study of local stability.

Introductory to intermediate, with room for more advanced exploration. The starting point uses familiar course models. Further work can develop a deeper study of stability and more general growth dynamics, according to participants' interests and the time available. The relevant growth economics can be introduced within the project.

Starting materials.

Oded Galor (2007), *Discrete Dynamical Systems*.

Lecture Notes Advanced Mathematics (September 2025).

Daron Acemoglu (2009), *Introduction to Modern Economic Growth* (linked prepublication draft).

Nancy L. Stokey and Robert E. Lucas, Jr., with Edward C. Prescott (1989), *Recursive Methods in Economic Dynamics*.

Possible directions.

- **From optimal decisions to trajectories.** Start with the optimal saving policy from the course's one-sector growth example and regard it as a map from current to next-period capital. Identify the economically relevant state space, steady states and trajectories. Explain how consumption is recovered, and distinguish the fixed point of the Bellman operator from a fixed point of the capital policy.

- **What guarantees convergence?** Study increasing maps and invariant intervals. Establish conditions under which trajectories are monotone and bounded, and show how continuity identifies their limits as fixed points. Apply these arguments to the growth model. Use its explicit solution as a check, then ask which conclusions remain available when the saving policy has no closed-form expression.

- **Local stability and economic persistence.** Introduce local asymptotic stability and prove the derivative criterion $|g'(k^*)| < 1$ using the mean value theorem. Compare local stability with attraction from all positive initial capital stocks. Investigate how production, productivity and patience affect steady-state capital and adjustment speed, and what the results imply for income differences between economies.

- **Can optimal growth cycle?** Use the policy supplied for the Weitzman two-sector model to study its second iterate and two-period paths. Examine why an increasing one-dimensional policy rules out nontrivial cycles, while the two-sector model allows them. Distinguish the existence of periodic paths from their stability, and interpret how the allocation of resources between sectors changes the dynamics.

- **Beyond the starting models.** With more time, investigate a broader class of saving policies using Lyapunov functions or other sufficient conditions for global stability. Alternatively, examine linear production and the possibility of sustained growth. More advanced work could extend the analysis to additional state variables, introducing the required mathematics progressively.

Expected outcome.

The project offers a way to learn dynamical systems through growth economics. It can develop an understanding of why some optimal paths converge, why others cycle, and how economic assumptions enter stability arguments. A theoretical report could connect selected proofs with a comparison of the one-sector and two-sector models, explaining their implications for saving, consumption and long-run development.

The inquiry can expand with the time devoted to it. A focused version could establish and interpret the basic convergence and stability results. A more extensive version could develop the comparison with cycles, investigate a general stability criterion, or study a richer growth model. The possible directions provide starting points for choosing how far to pursue the mathematics and the economics.