# Spin, Wobble, Flip:
# The Rigid Body in Three Dimensions

*BAMS project - Physics I*


## Abstract.

Hold a racket by its handle with the face horizontal, toss it so that it turns once head over handle, and catch it again. However carefully you throw, the face that was looking up now looks down: on its way round, the racket has added a half twist of its own. Spun about its handle, or flat like a frisbee, it behaves. In 1985 the cosmonaut Vladimir Dzhanibekov saw the same thing in orbit: a spinning wing nut that turned itself over every few seconds, with no force acting on it.


The explanation starts from a fact that stays hidden as long as a body turns about a fixed axle: for a body that is free to rotate in space, the angular momentum and the angular velocity usually point in *different* directions. The moment of inertia is no longer a number but a $3\times3$ symmetric matrix, the **inertia tensor**. Most of what is surprising about spinning objects, from the wobble of a thrown plate to the flip of the racket in Figure 1, comes from this matrix.

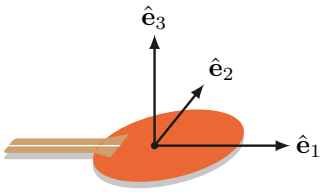



**Figure 1.** A racket and its three principal axes. A spin about $\hat{\mathbf{e}}_1$ or $\hat{\mathbf{e}}_3$ is steady. A spin about $\hat{\mathbf{e}}_2$ never is.

The project follows the free rotation of a rigid body from its equations to its behaviour: why every body has three special axes, why a spin about one of them is always unstable, and how the instability shows up in a simulation and in a real object thrown in the air.

## What to do.

The common goal is the **tennis racket theorem**: a proof, a simulation that reproduces it, and a measurement that confirms it. Steps 1–3 are the theory, steps 4–5 test it.

1. **The inertia tensor.** *Describe the rotation of a body about an axis of any direction.*
   - Every point of the body moves with velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Start from $\mathbf{L} = \sum m\, \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$ and show that $\mathbf{L} = I\boldsymbol{\omega}$ and $T = \frac{1}{2}\, \boldsymbol{\omega} \cdot I\boldsymbol{\omega}$, where $I$ is a $3 \times 3$ matrix that depends only on how the mass is distributed.
   - $I$ is symmetric. Use the spectral theorem to show that every body has three perpendicular *principal axes*, and interpret the eigenvalues $I_1 \le I_2 \le I_3$. When is $\mathbf{L}$ parallel to $\boldsymbol{\omega}$?
   - Compute $I$ for a few bodies: a rectangular plate, a cylinder, a racket made of a disc and a rod.
2. **Euler's equations.** *Find the equations of motion of a body with no external torque.*
   - In a frame attached to the body, with axes along the principal axes, $I$ is constant and diagonal, but the frame itself rotates. Find how the rate of change of a vector differs between the two frames, and apply it to $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$.
   - Find all the steady rotations. Solve the case of two equal moments: the wobble of a thrown plate, and the slow wandering of the Earth's axis.
3. **The tennis racket theorem.** *Decide which steady rotations survive a small disturbance.*
   - Perturb a steady spin about each principal axis and linearise. Which misalignments grow, and which only oscillate?
   - Find two conserved quantities and use them to draw the paths of $\mathbf{L}$ seen from the body (Figure 2a). Explain why the flip repeats at regular intervals.

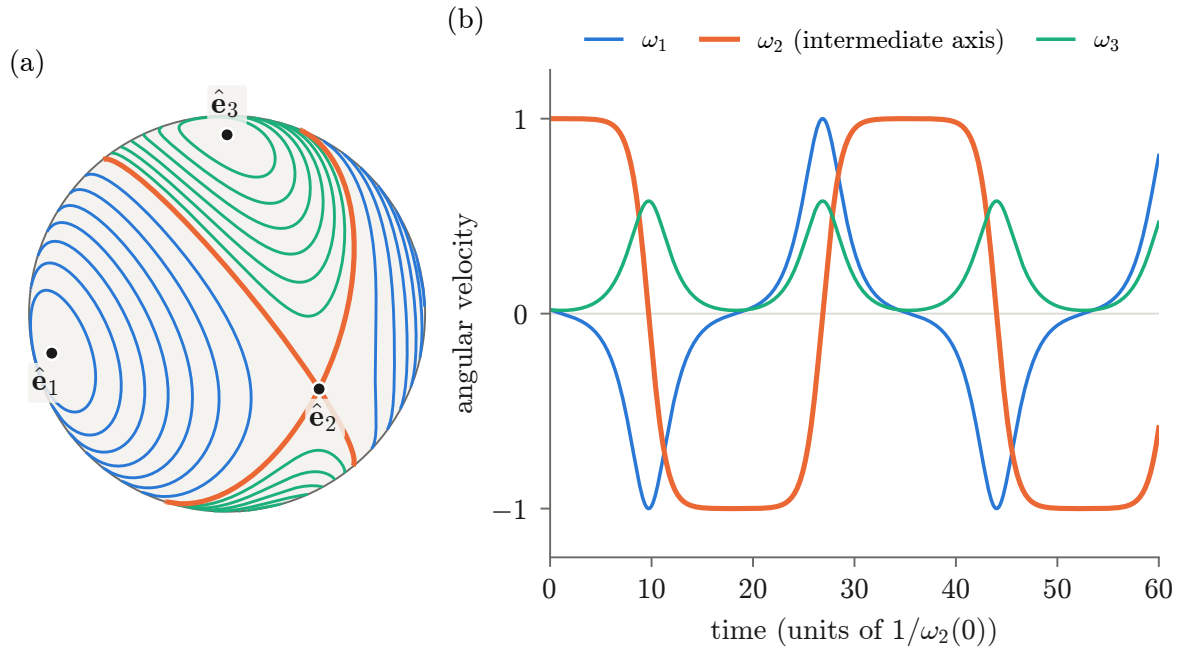



**Figure 2.** A free body with $I_1 : I_2 : I_3 = 1 : 2 : 3$. (a) The angular momentum seen from the body moves on a sphere, along curves of constant energy. Curves close around $\hat{\mathbf{e}}_1$ (blue) and $\hat{\mathbf{e}}_3$ (green). At $\hat{\mathbf{e}}_2$ two branches of the same curve (orange) cross. (b) A spin that starts almost exactly along $\hat{\mathbf{e}}_2$.

4. **Simulation.** *Solve Euler's equations numerically and watch the flip.*

   - Write a fourth-order Runge–Kutta integrator from scratch (about twenty lines of Python, no library solvers) and test it on the conserved quantities of step 3.
   - Reproduce Figure 2(b). How does the time between two flips depend on the initial misalignment?

5. **Experiment.** *See the theorem in a real object.*

   A smartphone has three different principal moments, and its gyroscope records $\boldsymbol{\omega}$ along the phone's own axes, which are the variables of Euler's equations. Spin one in the air (over something soft) and compare the data with the simulation.

6. **Further directions.**

   - Follow the orientation of the body in the room, with rotation matrices or quaternions, and animate the flip.
   - Add a torque: the precession of a gyroscope or of a spinning top.
   - Add a small loss of energy, and find out why only one of the three steady spins survives in the long run.

## Why rigid bodies?

The first American satellite, Explorer 1, was launched in 1958 spinning about its long axis, the one with the smallest moment of inertia. Soon after launch it was tumbling end over end: its flexible antennas dissipated a little energy, and a free body that loses energy at constant angular momentum ends up rotating about the axis with the *largest* moment. Spacecraft have been designed with this in mind ever since, and the same equations are used today to control the attitude of satellites and drones.

The ideas also appear far from mechanics. The principal axes of a body are the eigenvectors of the covariance matrix of its mass distribution, so finding them is the same computation as a principal component analysis. Testing the stability of a solution by looking at small perturbations is the standard first step in the study of any dynamical system. The orientation of an object in space, which every phone, game controller and robot has to estimate all the time, is described by rotation matrices and quaternions.

## Expected outcome.

A short illustrated report with:

- the derivation of Euler's equations and two proofs of the tennis racket theorem, one linear and one from the conserved quantities;
- a simulation, tested on its conserved quantities;
- measurements from a real spinning object, compared with the simulation.

Theory, code and experiment are separate strands, so the work divides naturally among participants.

## Prerequisites / Difficulty.

- Rotation about a fixed axis: moment of inertia, angular momentum, torque.
- Linear algebra: symmetric matrices, eigenvalues and eigenvectors.
- Basic Python and NumPy. No experience with numerical methods is needed.

**Intermediate.** Steps 1–5 need only Physics I and linear algebra. The further directions are open-ended and more demanding.

## Starting materials.

**Main reference.** D. Kleppner and R. Kolenkow, *An Introduction to Mechanics*. The chapter on rigid body motion (Chapter 8 in the second edition, Chapter 7 in the first) continues the textbook of Physics I into three dimensions.

**Theory and problems.** David Morin (2008). *Introduction to Classical Mechanics*, Chapter 9, with the inertia tensor (§9.2), principal axes (§9.3), Euler's equations and many solved problems. For a more mathematical treatment, David Tong, Lectures on Classical Dynamics, Chapter 3, free online.

**A first read.** R. Feynman, *The Feynman Lectures on Physics*, Vol. I, Chapter 20, Rotation in Space, free online. The video *The Bizarre Behavior of Rotating Bodies* by Veritasium (2019) shows the effect and tells the story of Dzhanibekov's wing nut.

**The theorem.** M. Ashbaugh, C. Chicone and R. Cushman (1991). The twisting tennis racket. *Journal of Dynamics and Differential Equations* 3, 67–85.

**The experiment.** phyphox, a free app developed at RWTH Aachen University that records the gyroscope of a phone and exports the data as a CSV file.

## Contacts.

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