

The Plateau Problem and the Birth of Geometric PDEs

Antonio De Rosa

Bocconi



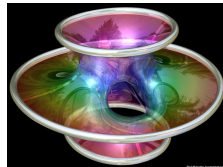
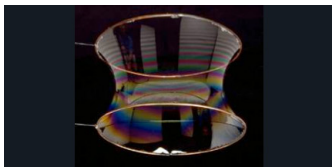
Funded by
the European Union



European Research Council
Established by the European Commission

Focus of the talk: Plateau problem

Existence and regularity of a surface with least area spanning a given boundary.



History: graphical case

Lagrange (1760) first posed the problem for graphs.

History: graphical case

Lagrange (1760) first posed the problem for graphs.

Consider a smooth $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, where Ω is bounded open set.

History: graphical case

Lagrange (1760) first posed the problem for graphs.

Consider a smooth $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, where Ω is bounded open set.

The **surface** is the graph

$$\Sigma = \{(x, y, u(x, y)) : (x, y) \in \Omega\}.$$

History: graphical case

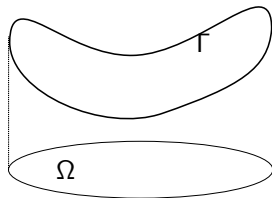
Lagrange (1760) first posed the problem for graphs.

Consider a smooth $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, where Ω is bounded open set.

The **surface** is the graph

$$\Sigma = \{(x, y, u(x, y)) : (x, y) \in \Omega\}.$$

The **boundary** is defined as $\Gamma := u(\partial\Omega)$.



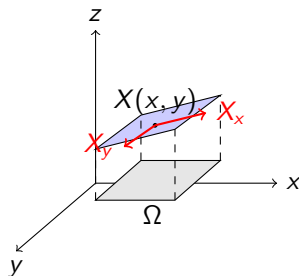
Area of the graph of a function

A parametrization for Σ is

$$X(x, y) = (x, y, u(x, y)).$$

Hence the tangent vectors are

$$X_x = (1, 0, u_x), \quad X_y = (0, 1, u_y).$$



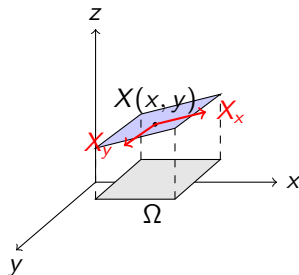
Area of the graph of a function

A parametrization for Σ is

$$X(x, y) = (x, y, u(x, y)).$$

Hence the tangent vectors are

$$X_x = (1, 0, u_x), \quad X_y = (0, 1, u_y).$$



The infinitesimal area element is

$$|X_x \times X_y| = \sqrt{1 + u_x^2 + u_y^2} = \sqrt{1 + |\nabla u|^2}.$$

Therefore,

$$A(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx \, dy.$$

First derivative test?

We want to minimize the area functional $A(u)$ over functions $u : \Omega \rightarrow \mathbb{R}$.

First derivative test?

We want to minimize the area functional $A(u)$ over functions $u : \Omega \rightarrow \mathbb{R}$.

Analogy with a finite dimensional space:

For $f : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$x \text{ is critical} \iff Df(x) = 0 \iff \left. \frac{d}{dt} f(x + tv) \right|_{t=0} = 0 \quad \forall v \in \mathbb{R}^n.$$

First derivative test?

We want to minimize the area functional $A(u)$ over functions $u : \Omega \rightarrow \mathbb{R}$.

Analogy with a finite dimensional space:

For $f : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$x \text{ is critical} \iff Df(x) = 0 \iff \left. \frac{d}{dt} f(x + tv) \right|_{t=0} = 0 \quad \forall v \in \mathbb{R}^n.$$

Infinite-dimensional version:

We perturb u in a direction φ :

$$u_t = u + t\varphi.$$

Then u is a critical point if

$$\left. \frac{d}{dt} A(u + t\varphi) \right|_{t=0} = 0 \quad \text{for all } \varphi.$$

Welcome to Calculus of Variations!

Courtesy of Euler and Lagrange

First variation of the area of a graph

Consider a variation

$$u_t = u + t\varphi, \quad \varphi \in C_c^\infty(\Omega),$$

so that

$$A(u_t) = \int_{\Omega} \sqrt{1 + |\nabla u + t\nabla\varphi|^2} \, dx \, dy.$$

First variation of the area of a graph

Consider a variation

$$u_t = u + t\varphi, \quad \varphi \in C_c^\infty(\Omega),$$

so that

$$A(u_t) = \int_{\Omega} \sqrt{1 + |\nabla u + t\nabla\varphi|^2} \, dx \, dy.$$

Differentiating at $t = 0$,

$$\frac{d}{dt}A(u_t)\Big|_{t=0} = \int_{\Omega} \frac{\nabla u \cdot \nabla\varphi}{\sqrt{1 + |\nabla u|^2}} \, dx \, dy.$$

First variation of the area of a graph

Consider a variation

$$u_t = u + t\varphi, \quad \varphi \in C_c^\infty(\Omega),$$

so that

$$A(u_t) = \int_{\Omega} \sqrt{1 + |\nabla u + t\nabla\varphi|^2} \, dx \, dy.$$

Differentiating at $t = 0$,

$$\left. \frac{d}{dt} A(u_t) \right|_{t=0} = \int_{\Omega} \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}} \, dx \, dy.$$

Using integration by parts and the compact support of φ ,

$$\int_{\Omega} \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}} = - \int_{\Omega} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \varphi.$$

First variation of the area of a graph

Consider a variation

$$u_t = u + t\varphi, \quad \varphi \in C_c^\infty(\Omega),$$

so that

$$A(u_t) = \int_{\Omega} \sqrt{1 + |\nabla u + t\nabla\varphi|^2} \, dx \, dy.$$

Differentiating at $t = 0$,

$$\left. \frac{d}{dt} A(u_t) \right|_{t=0} = \int_{\Omega} \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}} \, dx \, dy.$$

Using integration by parts and the compact support of φ ,

$$\int_{\Omega} \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}} = - \int_{\Omega} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \varphi.$$

Hence

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0.$$

Generalization to every dimension

Consider the problem:

$$\min \left\{ \int_{\Omega} \sqrt{1 + |\nabla v|^2} \quad v \in \text{Lip}(\Omega) \quad v = u_0 \text{ on } \partial\Omega \right\}, \quad \Omega \subset \mathbb{R}^{n-1}$$

Theorem

A minimizer u satisfies

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } \Omega \quad \text{minimal surface equation.}$$

Generalization to every dimension

Consider the problem:

$$\min \left\{ \int_{\Omega} \sqrt{1 + |\nabla v|^2} \quad v \in \text{Lip}(\Omega) \quad v = u_0 \text{ on } \partial\Omega \right\}, \quad \Omega \subset \mathbb{R}^{n-1}$$

Theorem

A minimizer u satisfies

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } \Omega \quad \text{minimal surface equation.}$$

Warning: This is not an existence result

Plateau and soap films

Plateau (1801–1883) - Belgian physicist fascinated by minimal surfaces

Performed experiments with soap films dipping wire frames in soap solution.

Plateau and soap films

Plateau (1801–1883) - Belgian physicist fascinated by minimal surfaces

Performed experiments with soap films dipping wire frames in soap solution.

Observation:

- Soap films naturally arrange themselves to minimize area.

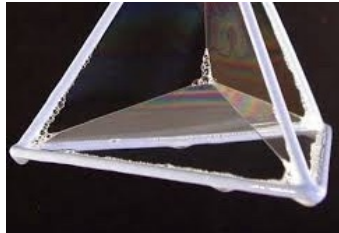
Plateau and soap films

Plateau (1801–1883) - Belgian physicist fascinated by minimal surfaces

Performed experiments with **soap films** dipping wire frames in soap solution.

Observation:

- Soap films naturally arrange themselves to minimize area.
- They form smooth surfaces, except for a structured singular set.



Plateau's Problem

Plateau's Problem

Given a boundary curve, find a surface of minimal area spanning it.

Plateau's laws

From experiments with soap films, Plateau observed:

- Surfaces are smooth and have zero **mean curvature**

Plateau's laws

From experiments with soap films, Plateau observed:

- Surfaces are smooth and have zero **mean curvature**
- Films meet in threes along curves at angles of 120°

Plateau's laws

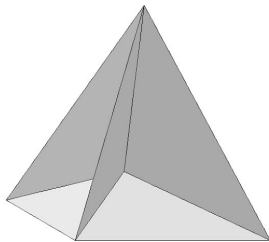
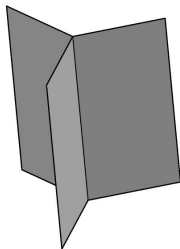
From experiments with soap films, Plateau observed:

- Surfaces are smooth and have zero **mean curvature**
- Films meet in threes along curves at angles of 120°
- These curves meet in fours with fixed angle at isolated points

Plateau's laws

From experiments with soap films, Plateau observed:

- Surfaces are smooth and have zero **mean curvature**
- Films meet in threes along curves at angles of 120°
- These curves meet in fours with fixed angle at isolated points



Mean curvature (intuition in 2D surfaces)

The mean curvature measures how a surface bends in space.

At each point of a surface, there are two principal curvatures: k_1, k_2 .

Mean curvature is their average:

$$H = \frac{k_1 + k_2}{2}.$$

Mean curvature (intuition in 2D surfaces)

The mean curvature measures how a surface bends in space.

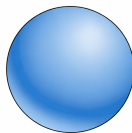
At each point of a surface, there are two principal curvatures: k_1, k_2 .

Mean curvature is their average:

$$H = \frac{k_1 + k_2}{2}.$$

Examples:

- Plane: $H = 0$ (no bending)
- Sphere: $H > 0$ (bends the same way in all directions)
- Catenoid: $H = 0$ (bends in opposite ways along two different directions)



Sphere: $H > 0$



Catenoid (minimal): $H = 0$

Parametric approach

Douglas (first Fields Medal in 1936) and Radó proposed the first proof of existence of solutions.

Parametric approach

Douglas (first Fields Medal in 1936) and Radó proposed the first proof of existence of solutions.

The **boundary** is a Jordan curve Γ , i.e. $\Gamma = \varphi(\mathbb{S}^1)$, φ injective and continuous.

Parametric approach

Douglas (first Fields Medal in 1936) and Radó proposed the first proof of existence of solutions.

The **boundary** is a Jordan curve Γ , i.e. $\Gamma = \varphi(\mathbb{S}^1)$, φ injective and continuous.

The **class of competitors** considered is

$$\{u \in C^0(\overline{\mathbb{D}^2}; \mathbb{R}^n) \cap C^1(\mathbb{D}^2; \mathbb{R}^n), u|_{\partial\mathbb{D}^2} \text{ homeomorphism}, \\ u|_{\mathbb{D}^2} \text{ immersion}, u(\partial\mathbb{D}^2) = \Gamma\}.$$

Parametric approach

Douglas (first Fields Medal in 1936) and Radó proposed the first proof of existence of solutions.

The **boundary** is a Jordan curve Γ , i.e. $\Gamma = \varphi(\mathbb{S}^1)$, φ injective and continuous.

The **class of competitors** considered is

$$\{u \in C^0(\overline{\mathbb{D}^2}; \mathbb{R}^n) \cap C^1(\mathbb{D}^2; \mathbb{R}^n), u|_{\partial\mathbb{D}^2} \text{ homeomorphism}, \\ u|_{\mathbb{D}^2} \text{ immersion}, u(\partial\mathbb{D}^2) = \Gamma\}.$$

The **area** of the surface generated by u is:

$$A(u) := \int_{\mathbb{D}^2} \left| \frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y} \right|.$$

Parametric approach

Douglas (first Fields Medal in 1936) and Radó proposed the first proof of existence of solutions.

The **boundary** is a Jordan curve Γ , i.e. $\Gamma = \varphi(\mathbb{S}^1)$, φ injective and continuous.

The **class of competitors** considered is

$$\{u \in C^0(\overline{\mathbb{D}^2}; \mathbb{R}^n) \cap C^1(\mathbb{D}^2; \mathbb{R}^n), u|_{\partial\mathbb{D}^2} \text{ homeomorphism}, \\ u|_{\mathbb{D}^2} \text{ immersion}, u(\partial\mathbb{D}^2) = \Gamma\}.$$

The **area** of the surface generated by u is:

$$A(u) := \int_{\mathbb{D}^2} \left| \frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y} \right|.$$

Theorem (Douglas and Radó, 1930): there exists a minimizer u .

Idea of the proof (Douglas)

Step 1: Relax the problem.

Instead of minimizing the area functional, we consider the Dirichlet energy

$$E(u) := \frac{1}{2} \int_{\mathbb{D}^2} |\nabla u|^2.$$

Key point: minimizing E is analytically easier (convex functional, weak compactness).

Idea of the proof (Douglas)

Step 1: Relax the problem.

Instead of minimizing the area functional, we consider the Dirichlet energy

$$E(u) := \frac{1}{2} \int_{\mathbb{D}^2} |\nabla u|^2.$$

Key point: minimizing E is analytically easier (convex functional, weak compactness).

Step 2: Reparametrization invariance of the area.

For every diffeomorphism $\psi : \mathbb{D}^2 \rightarrow \mathbb{D}^2$,

$$A(u) = A(u \circ \psi).$$

Interpretation: the area depends only on the surface, not on its parametrization.

Idea of the proof (Douglas)

Step 3: Energy dominates area. For every admissible map u , we have

$$A(u) \leq E(u).$$

Idea of the proof (Douglas)

Step 3: Energy dominates area. For every admissible map u , we have

$$A(u) \leq E(u).$$

Indeed, pointwise:

$$\left| \frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y} \right| \leq \left| \frac{\partial u}{\partial x} \right| \left| \frac{\partial u}{\partial y} \right|$$

and by the arithmetic–geometric inequality,

$$\left| \frac{\partial u}{\partial x} \right| \left| \frac{\partial u}{\partial y} \right| \leq \frac{1}{2} \left(\left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial y} \right|^2 \right).$$

Integrating over $\mathbb{D}^2$ gives $A(u) \leq E(u)$.

When do we have equality?

Equality case: $A(u) = E(u)$ if and only if u is *conformal*, i.e.

$$\left| \frac{\partial u}{\partial x} \right| = \left| \frac{\partial u}{\partial y} \right|, \quad \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 0.$$

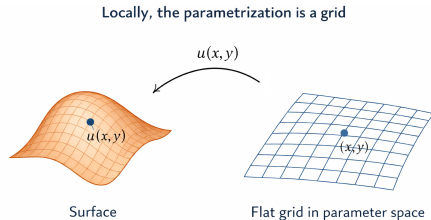
When do we have equality?

Equality case: $A(u) = E(u)$ if and only if u is *conformal*, i.e.

$$\left| \frac{\partial u}{\partial x} \right| = \left| \frac{\partial u}{\partial y} \right|, \quad \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 0.$$

Interpretation:

- the parametrization preserves angles,
- the metric is isotropic: $|\partial_x u|^2 = |\partial_y u|^2$.



Idea of the proof (Douglas)

Step 4: Uniformization of parametrizations

Lichtenstein theorem:

For every admissible map u , there exists a diffeomorphism $\sigma : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ such that

$$u \circ \sigma \quad \text{is conformal.}$$

Idea: we can always reparametrize the surface so that it becomes conformal.

Conclusion of the proof (Douglas)

Let $\bar{u}$ be a minimizer of the Dirichlet energy:

$$E(\bar{u}) \leq E(u) \quad \text{for all admissible } u.$$

Conclusion of the proof (Douglas)

Let $\bar{u}$ be a minimizer of the Dirichlet energy:

$$E(\bar{u}) \leq E(u) \quad \text{for all admissible } u.$$

For any competitor u , using Lichtenstein theorem:

$$A(u) = A(u \circ \sigma) = E(u \circ \sigma).$$

Conclusion of the proof (Douglas)

Let $\bar{u}$ be a minimizer of the Dirichlet energy:

$$E(\bar{u}) \leq E(u) \quad \text{for all admissible } u.$$

For any competitor u , using Lichtenstein theorem:

$$A(u) = A(u \circ \sigma) = E(u \circ \sigma).$$

Hence

$$A(u) = E(u \circ \sigma) \geq E(\bar{u}) \geq A(\bar{u}).$$

Conclusion: $\bar{u}$ minimizes the area, hence solves Plateau's problem.

Conclusion of the proof (Douglas)

Let $\bar{u}$ be a minimizer of the Dirichlet energy:

$$E(\bar{u}) \leq E(u) \quad \text{for all admissible } u.$$

For any competitor u , using Lichtenstein theorem:

$$A(u) = A(u \circ \sigma) = E(u \circ \sigma).$$

Hence

$$A(u) = E(u \circ \sigma) \geq E(\bar{u}) \geq A(\bar{u}).$$

Conclusion: $\bar{u}$ minimizes the area, hence solves Plateau's problem.

Limitations of the method: In higher dimensions, Lichtenstein theorem is false.

Birth of Geometric Measure Theory

New idea: enlarge the class of competitors.

Instead of smooth parametrized surfaces, consider *weak objects*:

- rectifiable measures,
- currents (De Giorgi, Federer–Fleming),
- varifolds (Almgren, Allard).

Birth of Geometric Measure Theory

New idea: enlarge the class of competitors.

Instead of smooth parametrized surfaces, consider *weak objects*:

- rectifiable measures,
- currents (De Giorgi, Federer–Fleming),
- varifolds (Almgren, Allard).

Philosophy:

lose parametrization, gain compactness and lower semicontinuity.

Birth of Geometric Measure Theory

New idea: enlarge the class of competitors.

Instead of smooth parametrized surfaces, consider *weak objects*:

- rectifiable measures,
- currents (De Giorgi, Federer–Fleming),
- varifolds (Almgren, Allard).

Philosophy:

lose parametrization, gain compactness and lower semicontinuity.

Geometric Measure Theory leads to general solutions of Plateau's problem in any dimension and codimension.

The direct method in the calculus of variations

Let (X, τ) be a topological space and $J : X \rightarrow \mathbb{R}$.

Goal: find a minimizer, i.e. $x \in X$ such that

$$J(x) \leq J(u) \quad \forall u \in X.$$

The direct method in the calculus of variations

Let (X, τ) be a topological space and $J : X \rightarrow \mathbb{R}$.

Goal: find a minimizer, i.e. $x \in X$ such that

$$J(x) \leq J(u) \quad \forall u \in X.$$

First observation: if $X \neq \emptyset$ (otherwise nothing to prove), there always exists a *minimizing sequence* $(x_n) \subset X$ such that

$$J(x_n) \longrightarrow \inf_{u \in X} J(u).$$

The direct method: scheme of the proof

Step 1: Minimizing sequence

Take $(x_n) \subset X$ such that

$$J(x_n) \rightarrow \inf_{x \in X} J(x).$$

The direct method: scheme of the proof

Step 1: Minimizing sequence

Take $(x_n) \subset X$ such that

$$J(x_n) \rightarrow \inf_{x \in X} J(x).$$

Step 2: Compactness

Show that there exists a subsequence (x_{n_k}) and $x \in X$ such that

$$x_{n_k} \xrightarrow{\tau} x.$$

The direct method: scheme of the proof

Step 1: Minimizing sequence

Take $(x_n) \subset X$ such that

$$J(x_n) \rightarrow \inf_{x \in X} J(x).$$

Step 2: Compactness

Show that there exists a subsequence (x_{n_k}) and $x \in X$ such that

$$x_{n_k} \xrightarrow{\tau} x.$$

Step 3: Lower semicontinuity

Show that J is lower semicontinuous with respect to τ , i.e.

$$J(x) \leq \liminf_{k \rightarrow \infty} J(x_{n_k}).$$

The direct method: scheme of the proof

Step 1: Minimizing sequence

Take $(x_n) \subset X$ such that

$$J(x_n) \rightarrow \inf_{x \in X} J(x).$$

Step 2: Compactness

Show that there exists a subsequence (x_{n_k}) and $x \in X$ such that

$$x_{n_k} \xrightarrow{\tau} x.$$

Step 3: Lower semicontinuity

Show that J is lower semicontinuous with respect to τ , i.e.

$$J(x) \leq \liminf_{k \rightarrow \infty} J(x_{n_k}).$$

Conclusion:

$$J(x) \leq \liminf J(x_{n_k}) = \inf_X J,$$

hence x is a minimizer.

A natural larger class: measures

Problem: classes of smooth or parametrized surfaces are not compact.

Idea: describe surfaces as *mass distributions*. Sampling points from these measures will approximate the original surfaces.

A natural larger class: measures

Problem: classes of smooth or parametrized surfaces are not compact.

Idea: describe surfaces as *mass distributions*. Sampling points from these measures will approximate the original surfaces.

Natural setting: Radon measures on $\mathbb{R}^n$.

- identify a surface Σ with a measure μ , where $\mu(A) = \text{Area}(\Sigma \cap A)$.

A natural larger class: measures

Problem: classes of smooth or parametrized surfaces are not compact.

Idea: describe surfaces as *mass distributions*. Sampling points from these measures will approximate the original surfaces.

Natural setting: Radon measures on $\mathbb{R}^n$.

- identify a surface Σ with a measure μ , where $\mu(A) = \text{Area}(\Sigma \cap A)$.

Advantage:

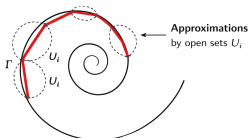
- weak compactness (Banach–Alaoglu type),
- lower semicontinuity in open sets.

Hausdorff measures

Definition (Hausdorff measure).

For $d \geq 0$ and $K \subset \mathbb{R}^n$, define

$$\mathcal{H}^d(K) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_i (\text{diam } U_i)^d : K \subset \bigcup_i U_i, \text{diam}(U_i) < \delta \right\}.$$

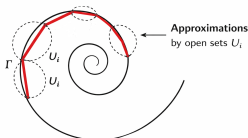


Hausdorff measures

Definition (Hausdorff measure).

For $d \geq 0$ and $K \subset \mathbb{R}^n$, define

$$\mathcal{H}^d(K) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_i (\text{diam } U_i)^d : K \subset \bigcup_i U_i, \text{diam}(U_i) < \delta \right\}.$$



Interpretation:

- $\mathcal{H}^1$ measures length,
- $\mathcal{H}^2$ measures surface area,
- for every $K \subset \mathbb{R}^n$, we define the measure $\mathcal{H}^d \llcorner K$

$$\mathcal{H}^d \llcorner K(E) = \mathcal{H}^d(K \cap E) = \text{Area}_d(K \cap E), \quad \forall E \subset \mathbb{R}^n.$$

Rectifiable sets and rectifiable measures

Definition (rectifiable set). A set $K \subset \mathbb{R}^n$ is d -rectifiable if there exists a countable family of C^1 maps

$$f_i : \mathbb{R}^d \rightarrow \mathbb{R}^n$$

such that

$$\mathcal{H}^d \left(K \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^d) \right) = 0.$$

Rectifiable sets and rectifiable measures

Definition (rectifiable set). A set $K \subset \mathbb{R}^n$ is d -rectifiable if there exists a countable family of C^1 maps

$$f_i : \mathbb{R}^d \rightarrow \mathbb{R}^n$$

such that

$$\mathcal{H}^d \left(K \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^d) \right) = 0.$$

Rectifiable measures. A Radon measure μ is called d -rectifiable if it can be written as

$$\mu = \theta \mathcal{H}^d \llcorner K,$$

where K is d -rectifiable and $\theta \in L^1_{\text{loc}}(\mathcal{H}^d \llcorner K)$, $\theta \geq 0$.

Rectifiable sets and rectifiable measures

Definition (rectifiable set). A set $K \subset \mathbb{R}^n$ is d -rectifiable if there exists a countable family of C^1 maps

$$f_i : \mathbb{R}^d \rightarrow \mathbb{R}^n$$

such that

$$\mathcal{H}^d \left(K \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^d) \right) = 0.$$

Rectifiable measures. A Radon measure μ is called d -rectifiable if it can be written as

$$\mu = \theta \mathcal{H}^d \llcorner K,$$

where K is d -rectifiable and $\theta \in L^1_{\text{loc}}(\mathcal{H}^d \llcorner K)$, $\theta \geq 0$.

Crucial fact. We can do first order calculus on rectifiable sets/measures since:

- tangent planes $T_x K$ exist for μ -a.e. x .

A GMT formulation of Plateau's problem

Let $H \subset \mathbb{R}^n$ be a closed set (the **boundary**).

We consider a **competitor class** $\mathcal{P}(H)$, closed by Lipschitz deformations, of relatively closed sets $K \subset \mathbb{R}^n \setminus H$.

A GMT formulation of Plateau's problem

Let $H \subset \mathbb{R}^n$ be a closed set (the **boundary**).

We consider a **competitor class** $\mathcal{P}(H)$, closed by Lipschitz deformations, of relatively closed sets $K \subset \mathbb{R}^n \setminus H$.

Plateau problem in $\mathcal{P}(H)$:

$$m_0 := \inf \{ \mathcal{H}^d(K) : K \in \mathcal{P}(H) \} .$$

Philosophy: minimize area among very general (possibly non-smooth) sets.

A GMT formulation of Plateau's problem

We want to apply the direct method.

Idea: We look at the sets K as measures

$$K \rightsquigarrow \mathcal{H}^d \llcorner K.$$

A GMT formulation of Plateau's problem

We want to apply the direct method.

Idea: We look at the sets K as measures

$$K \rightsquigarrow \mathcal{H}^d \llcorner K.$$

We get **compactness** (in the space of measures) and **lower semicontinuity** via standard functional analysis.

Warning: the minimizer we get is a priori just a measure.

Main result

Theorem (De Philippis, D. R., Ghiraldin, 2014)

Assume $m_0 < \infty$ and let $(K_j) \subset \mathcal{P}(H)$ be a minimizing sequence.

Existence: up to subsequences, the measures

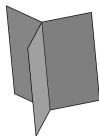
$$\mu_j := \mathcal{H}^d \llcorner K_j \rightharpoonup^* \mu = \mathcal{H}^d \llcorner K \quad \text{in } \mathbb{R}^n \setminus H,$$

where K is a d -rectifiable set and $\liminf_j \mathcal{H}^d(K_j) \geq \mathcal{H}^d(K)$.

Regularity: K is a real analytic submanifold outside a relatively closed set $\Sigma \subset K$ with $\dim_{\mathcal{H}}(\Sigma) \leq d - 1$.

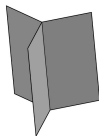
Regularity is sharp without making further assumptions on the class.

If $n = 3$, $d = 2$

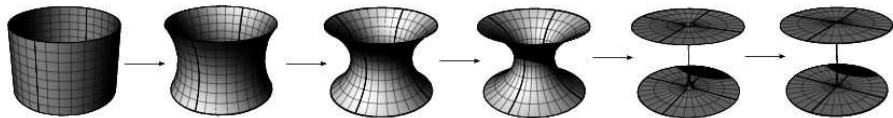


Regularity is sharp without making further assumptions on the class.

If $n = 3$, $d = 2$

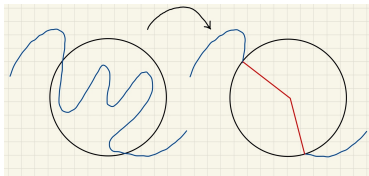


Also, take as boundary two circles far enough and a class of competitors closed by Lipschitz deformation containing catenoids.



The line between the two disks is a **singular set of dimension 1**.

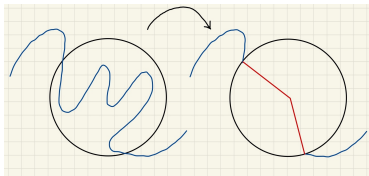
Monotonicity formula



Fix $x \in K$ and $r > 0$. Use a **cone competitor** in $B_{x,r}$:

$$f(r) := \mathcal{H}^d(K \cap B_{x,r}) \leq \mathcal{H}^d(\text{Cone} \cap B_{x,r}) = \frac{r}{d} \mathcal{H}^{d-1}(K \cap \partial B_{x,r}) = \frac{r}{d} f'(r)$$

Monotonicity formula

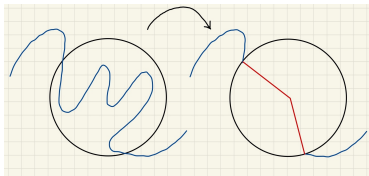


Fix $x \in K$ and $r > 0$. Use a **cone competitor** in $B_{x,r}$:

$$f(r) := \mathcal{H}^d(K \cap B_{x,r}) \leq \mathcal{H}^d(\text{Cone} \cap B_{x,r}) = \frac{r}{d} \mathcal{H}^{d-1}(K \cap \partial B_{x,r}) = \frac{r}{d} f'(r)$$

$$\Rightarrow \frac{d}{dr} [\log(f(r)) - d \log(r)] \geq 0.$$

Monotonicity formula



Fix $x \in K$ and $r > 0$. Use a **cone competitor** in $B_{x,r}$:

$$f(r) := \mathcal{H}^d(K \cap B_{x,r}) \leq \mathcal{H}^d(\text{Cone} \cap B_{x,r}) = \frac{r}{d} \mathcal{H}^{d-1}(K \cap \partial B_{x,r}) = \frac{r}{d} f'(r)$$

$$\Rightarrow \frac{d}{dr} [\log(f(r)) - d \log(r)] \geq 0.$$

Hence:

$$\frac{d}{dr} \frac{f(r)}{r^d} \geq 0.$$

From existence to regularity

The direct method gives a limiting object μ , which *a priori* is only a measure.

Problem: a measure has no obvious PDE (Euler–Lagrange minimal surface equation) as mass is preserved.

From existence to regularity

The direct method gives a limiting object μ , which *a priori* is only a measure.

Problem: a measure has no obvious PDE (Euler–Lagrange minimal surface equation) as mass is preserved.

To study the **regularity** of minimizers, we need a framework that:

- keeps track of both *mass* and *tangent directions*,
- allows us to write a weak version of the PDE (minimal surface equation).

This leads naturally to the notion of a **varifold**.

What is a varifold?

Let $G(n, d)$ be the Grassmannian of unoriented d -dimensional planes in $\mathbb{R}^n$.

A d -varifold in $\mathbb{R}^n$ is a Radon measure

$$V \quad \text{on } \mathbb{R}^n \times G(n, d).$$

What is a varifold?

Let $G(n, d)$ be the Grassmannian of unoriented d -dimensional planes in $\mathbb{R}^n$.

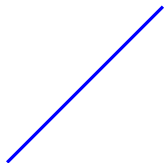
A **d -varifold** in $\mathbb{R}^n$ is a Radon measure

$$V \quad \text{on } \mathbb{R}^n \times G(n, d).$$

Interpretation:

- $x \in \mathbb{R}^n$ tells us *where* the surface is,
- $S \in G(n, d)$ tells us its *tangent plane*.

So a varifold records both position and infinitesimal geometry.



What is a varifold?

Let $G(n, d)$ be the Grassmannian of unoriented d -dimensional planes in $\mathbb{R}^n$.

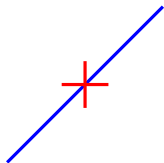
A **d -varifold** in $\mathbb{R}^n$ is a Radon measure

$$V \quad \text{on } \mathbb{R}^n \times G(n, d).$$

Interpretation:

- $x \in \mathbb{R}^n$ tells us *where* the surface is,
- $S \in G(n, d)$ tells us its *tangent plane*.

So a varifold records both position and infinitesimal geometry.



What is a varifold?

Let $G(n, d)$ be the Grassmannian of unoriented d -dimensional planes in $\mathbb{R}^n$.

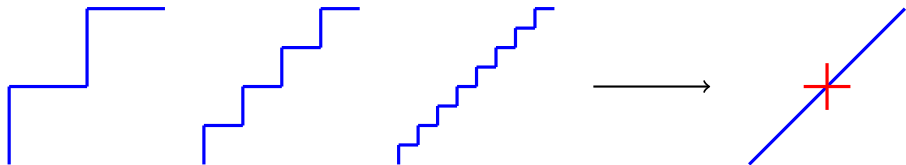
A **d -varifold** in $\mathbb{R}^n$ is a Radon measure

$$V \quad \text{on } \mathbb{R}^n \times G(n, d).$$

Interpretation:

- $x \in \mathbb{R}^n$ tells us *where* the surface is,
- $S \in G(n, d)$ tells us its *tangent plane*.

So a varifold records both position and infinitesimal geometry.



First variation of a varifold

Let $X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ be a vector field, and let $\{\Phi_t\}$ be its flow.

We deform the varifold by push-forward $(\Phi_t)_\# V$:

$$(\Phi_t)_\# V(\varphi) = \int_{\mathbb{R}^n \times G(n,d)} \varphi(\Phi_t(x), D\Phi_t(x) \cdot S) J_S \Phi_t(x) dV(x, S).$$

First variation of a varifold

Let $X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ be a vector field, and let $\{\Phi_t\}$ be its flow.

We deform the varifold by push-forward $(\Phi_t)_\# V$:

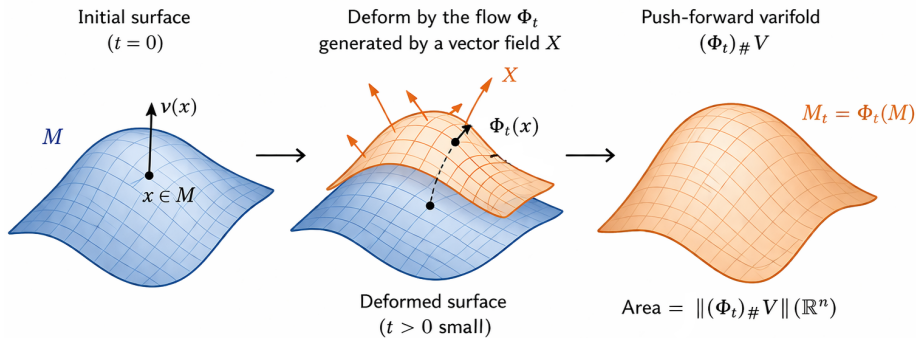
$$(\Phi_t)_\# V(\varphi) = \int_{\mathbb{R}^n \times G(n,d)} \varphi(\Phi_t(x), D\Phi_t(x) \cdot S) J_S \Phi_t(x) dV(x, S).$$

The **first variation** of V in the direction X is

$$\delta V(X) := \left. \frac{d}{dt} \right|_{t=0} \|(\Phi_t)_\# V\|(\mathbb{R}^n).$$

It measures the infinitesimal change of area under the deformation generated by X .

First variation of a smooth surface



$V = \mathcal{H}^d \llcorner M$
is the varifold of M

Φ_t is the flow of X :
 $\frac{d}{dt} \Phi_t(x) = X(\Phi_t(x)), \quad \Phi_0 = \text{Id}$

The first variation in direction X is
 $\delta V(X) = \left. \frac{d}{dt} \right|_{t=0} \|(\Phi_t)_\# V\|(\mathbb{R}^n)$

Formula for the first variation

For a general varifold V , one has

$$\delta V(X) = \int_{\mathbb{R}^n \times G(n,d)} \operatorname{div}_S X(x) dV(x, S),$$

where $\operatorname{div}_S X(x)$ denotes the tangential divergence of X along the plane S .

Formula for the first variation

For a general varifold V , one has

$$\delta V(X) = \int_{\mathbb{R}^n \times G(n,d)} \operatorname{div}_S X(x) dV(x, S),$$

where $\operatorname{div}_S X(x)$ denotes the tangential divergence of X along the plane S .

If V is associated with a smooth surface M , this becomes

$$\delta V(X) = \int_M \operatorname{div}_{T_x M} X d\mathcal{H}^d.$$

Stationary varifolds

A varifold V is called **stationary** if

$$\delta V(X) = 0 \quad \forall X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n).$$

This is the weak form of the minimal surface PDE.

Stationary varifolds

A varifold V is called **stationary** if

$$\delta V(X) = 0 \quad \forall X \in C_c^1(\mathbb{R}^n; \mathbb{R}^n).$$

This is the weak form of the minimal surface PDE.

For smooth surfaces, stationarity is equivalent to

$$H = 0,$$

that is, the surface is minimal.

Mean curvature in weak form

If the first variation is absolutely continuous with respect to $\|V\|$, we can write

$$\delta V(X) = - \int X \cdot H d\|V\|,$$

for some vector field $H \in L^1_{\text{loc}}(\|V\|; \mathbb{R}^n)$.

Mean curvature in weak form

If the first variation is absolutely continuous with respect to $\|V\|$, we can write

$$\delta V(X) = - \int X \cdot H d\|V\|,$$

for some vector field $H \in L^1_{\text{loc}}(\|V\|; \mathbb{R}^n)$.

This vector field H is the **generalized mean curvature** of the varifold.

Thus, even when the surface is not smooth, varifolds allow us to define mean curvature and to study a geometric PDE.

Mean curvature of a graph

For a smooth graph

$$M = \{(x, u(x)) : x \in \Omega\},$$

where $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$, the mean curvature is given by

$$H = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

Mean curvature of a graph

For a smooth graph

$$M = \{(x, u(x)) : x \in \Omega\},$$

where $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$, the mean curvature is given by

$$H = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

Therefore the PDE reads:

$$H = 0 \quad \Longleftrightarrow \quad \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

Thank you!