Donuts, Bottles and Other Monsters: An Atlas of Low-Dimensional Manifolds

BAMS project - Algebra and Geometry

Abstract.

From where we stand, the Earth looks flat. Only when we consider it as a whole do we discover that it is a sphere, and that someone who keeps walking straight ahead will eventually come back to where they started. A **manifold** is a space that behaves in the same way. Around each of its points it looks like a small piece of ordinary Euclidean space, but taken as a whole it can curve, close up on itself, or twist in ways that a flat space never could.

The simplest examples are surfaces, such as the sphere or the torus, the surface of a donut. Others are stranger. The Klein bottle is a surface with only one side, and it cannot be built in our three-dimensional space without passing through itself. One dimension higher we find the three-dimensional sphere, which can be filled entirely by circles, any two of them linked like the rings of a chain (Figure 1).

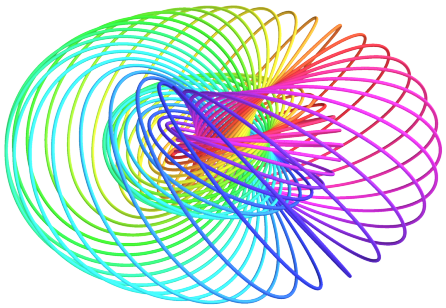


Figure 1. The Hopf fibration. The three-dimensional sphere is filled by circles, one for each point of the ordinary sphere, and any two of them are linked. Here it is projected into ordinary space.

The aim of the project is to get to know a small collection of these spaces, a bestiary of sorts, and to use it to approach a question that runs through all of geometry and topology: *when should two spaces be considered the same, and how can we prove that two of them are different?* For surfaces the question has a complete and very satisfying answer. Every closed surface turns out to be a sphere, a donut with some number of holes, or one of their one-sided relatives, and each of them carries a natural geometry, which is either round, flat or hyperbolic. In three dimensions the picture becomes much richer, and in four dimensions the story ends with a surprise: one can prove that no algorithm is able to decide whether two manifolds are the same.

The project is also a first introduction to three subjects that appear throughout mathematics. Topology gives a precise meaning to the idea of two spaces being the same. Group theory, the mathematics of symmetry, is needed both to build our spaces and to tell them apart. Differential geometry, finally, brings back lengths, angles and curvature, and explains why each surface has the geometry it has.

What we will do.

Everyone in the project works towards the same goal, which is to understand and prove the **classification of closed surfaces**. We get there in a few steps, starting with the bits of topology and group theory that are needed even to state the result, and once the theorem is proved we look at what it says about the geometry of surfaces. If time allows, we will end with a look at what happens in three dimensions.

1. **Spaces, gluing and symmetry.** We begin by learning the basic language. From topology we need the notions of topological space, continuous map and homeomorphism, the last one being the precise meaning of two spaces being "the same". From algebra we need the definition of a group and of a group acting on a space. The two ideas meet in the construction behind almost all of our examples, which is gluing. Identifying the opposite sides of a square produces a torus, which can also be described as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ of the plane by the integer translations. In the same way, identifying each point of a sphere with its opposite point produces the projective plane $S^2/\{\pm 1\}$.

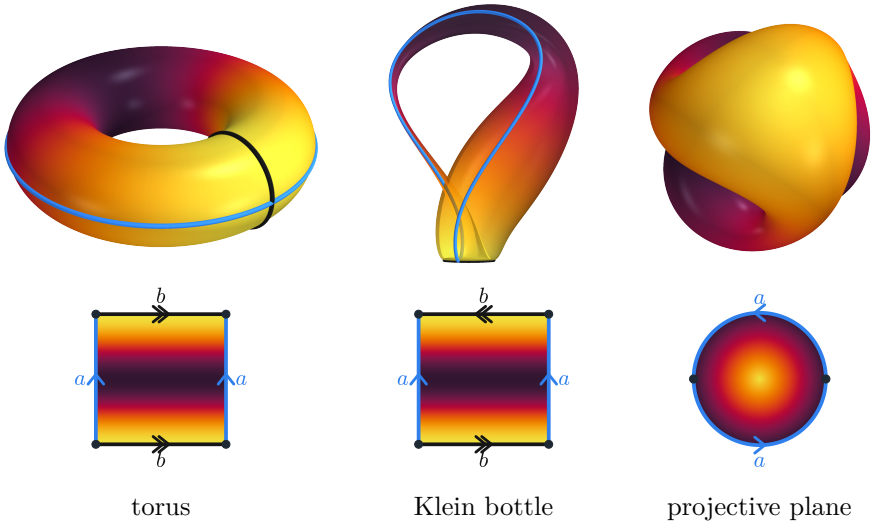
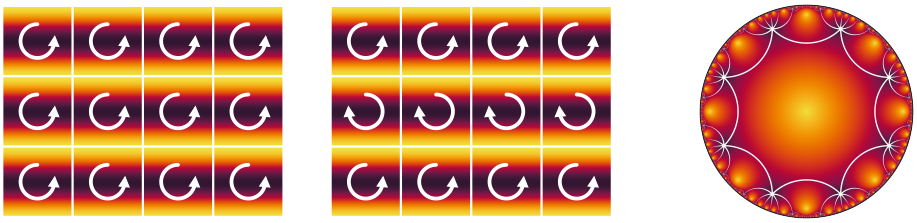


Figure 2. Three surfaces and the polygons that define them. Edges with the same letter are glued so that the arrows match, and the coloured curves on each surface show where the edges end up. The Klein bottle (drawn translucent) and the projective plane cannot sit in ordinary space without crossing themselves, so the self-intersections in the pictures only come from drawing them in three dimensions.

2. **The classification of surfaces.** This is the main part of the project. Starting from polygons whose edges are glued in pairs, as in Figure 2, a cut-and-paste argument shows that every closed surface is a sphere, a donut with some number of holes, or one of their one-sided relatives. To be sure that the surfaces on this list are really all different, we then need properties that cannot change when a space is deformed. The Euler characteristic, the number $V - E + F$ familiar from polyhedra, is the first example. A more powerful one is the fundamental group, which records the loops on a surface that cannot be shrunk to a point, and here group theory becomes really useful: for the torus, for instance, it is $\mathbb{Z}^2$.
3. **Curvature and geometry.** Once we know which surfaces exist, we can ask what shape they can take. Here differential geometry enters, with the curvature of a surface and the Gauss-Bonnet theorem, which says that the total curvature of a closed surface depends only on its Euler characteristic. It turns out that the sphere is naturally round, the torus is naturally flat, being a quotient of the Euclidean plane, and every surface with two or more holes is naturally hyperbolic, a quotient of the hyperbolic plane in the same way. Figure 3

shows what these three kinds of world look like from the inside.



(a) torus (b) Klein bottle (c) surface with two holes

Figure 3. What an inhabitant would see: copy after copy of the same polygon, with circular arrows marking a sense of rotation. (a) On the torus all copies look alike. (b) On the Klein bottle every other row is mirror-reversed and its arrows turn the other way, which is how an inhabitant would discover that the world has only one side. (c) A surface with two holes unrolls onto the hyperbolic plane, tiled by octagons.

4. **A step into three dimensions (if time allows).** Once surfaces are settled, we can turn to the three-dimensional sphere of Figure 1, which we can think of as the unit sphere in $\mathbb{C}^2$. It can be described using quaternions, a four-dimensional relative of the complex numbers, and this description leads directly to the Hopf fibration and to the fact that the set of all rotations of ordinary space is itself a three-dimensional manifold, obtained from the three-dimensional sphere by identifying opposite points. We would end with an informal discussion of why the classification cannot be carried on in four dimensions.

Why manifolds?

All the spaces in this project have dimension two or three, but that is mostly a matter of convenience: these are the ones we can draw, and the ones where our intuition works best. In general a manifold can have any dimension, and it does not need to sit inside any space we can picture. The set of all rotations of $\mathbb{R}^n$, or the set of all lines through the origin in $\mathbb{R}^n$, are manifolds too, of dimension that grows with n, and the ideas we develop for surfaces apply to them as well. Starting from low dimensions is a way to build intuition that can then be carried much further, and there is a lot more to explore beyond what we will see.

Manifolds are among the basic objects of modern mathematics, and anyone who goes on studying it will meet them again in geometry, in differential equations and in physics, where general relativity describes spacetime itself as a curved four-dimensional manifold.

They also appear in more concrete settings. The possible positions of a robot arm with two rotating joints form a torus, and the possible orientations of an object in space, which robotics and computer graphics handle all the time, form one of the three-dimensional manifolds of the last part. In machine learning, it is common to assume that high-dimensional data, such as images, lie close to a manifold of much smaller dimension.

Expected outcome.

By the end of the project, participants will have learned the basic language of topology, group theory and differential geometry on concrete examples, and will have proved a complete classification theorem from start to finish.

The work will be collected in a short **illustrated atlas** of the spaces met along the way. Each entry describes one space: how it is built, how it can be recognised, which geometry it carries, and what it looks like. Every participant can take care of one or two entries, so that the atlas becomes a joint piece of work.

Prerequisites / Difficulty.

- Linear algebra and the first analysis courses: matrices, continuity, open sets and compactness in $\mathbb{R}^n$.
- No previous knowledge of topology, group theory or differential geometry is needed. We start from the definitions, including a short introduction to groups.
- Some Python is useful, but not required, for drawing the spaces we study.

The core of the project, on surfaces, is accessible from the first year. The final part on three-dimensional spaces, which we will reach if time allows, is more demanding.

Starting materials.

Main reference. John M. Lee (2011). Introduction to Topological Manifolds, 2nd ed. Chapters 2 to 6 cover everything we need for the classification of surfaces, and chapters 7 and 8 introduce the fundamental group.

Groups. M. A. Armstrong (1988). Groups and Symmetry, in particular the chapters on subgroups, quotients and group actions.

Curves and surfaces. Kristopher Tapp (2016). Differential Geometry of Curves and Surfaces, a very readable and beautifully illustrated introduction that goes all the way to the Gauss-Bonnet theorem. Andrew Pressley (2010). Elementary Differential Geometry, 2nd ed., which also covers hyperbolic geometry.

A first read. Jeffrey Weeks, *The Shape of Space*, a tour of two- and three-dimensional spaces with almost no prerequisites.

Two short articles. George K. Francis and Jeffrey R. Weeks (1999). Conway's ZIP proof, a proof of the classification of surfaces in seven pages. David W. Lyons (2003). An elementary introduction to the Hopf fibration, for the part on three dimensions.

Contacts.

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