The Laziest Path:
From Least Action to the Double Pendulum

BAMS project - Physics I

Abstract.

Throw a ball to a friend. Of all the ways in which it could travel from your hand to theirs in the same time, why does it choose that one? Newton's answer is local: at every instant the force tells the ball how to change its velocity. There is a second answer, found about a century later, that sounds completely different. Give every conceivable path a number, its **action**, by adding up along the path the difference between kinetic and potential energy. The path that the ball really follows is the one that makes this number as small as possible (Figure 2).

This is the **principle of least action**, and it turns mechanics into a problem of optimisation. Its practical form is the Lagrangian method: a single function, $L = T - V$, written in whatever coordinates are convenient, gives all the equations of motion. There are no free-body diagrams, no tensions or normal forces, no components to project.

The principle also explains where conservation laws come from. Every symmetry of a system gives a conserved quantity: momentum is conserved because no place in space is special, energy because no moment in time is. This is Noether's theorem, one of the most useful ideas in physics.

The **double pendulum**, a pendulum hanging from another pendulum, puts all of this to work. Its equations of motion take a few lines to derive with a Lagrangian, but they have no solution in closed form, and the motion they describe is chaotic: two pendulums released from almost the same position soon have nothing in common (Figure 1).

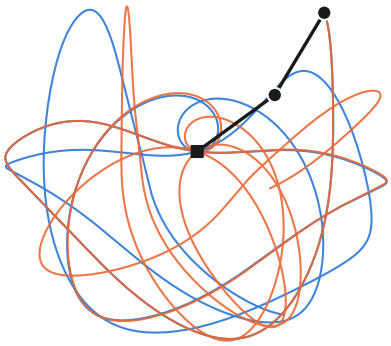


Figure 1. The lower bob of two double pendulums released from rest, the second (orange) with its lower arm turned by one extra thousandth of a radian. The black arms show the starting position.

What to do.

The common goal is the double pendulum: its equations from the principle of least action, and its motion from a simulation written from scratch. Steps 1–3 build the method, steps 4–5 apply it.

1. **The calculus of variations.** *Find the curve that makes a functional as small as possible.*

   - A *functional* assigns a number to a whole curve: its length, the time needed to slide along it. Perturb a curve with fixed endpoints and derive the condition that a stationary curve must satisfy, the Euler–Lagrange equation.
   - Use it on three classical problems: the shortest path in the plane, the soap film between two rings, and the brachistochrone, the slide of fastest descent.

2. **Lagrangian mechanics.** *Replace forces with a single function.*

   - Define the action $S = \int L \, dt$ with $L = T - V$, and check that for a particle the Euler–Lagrange equation is Newton's second law. Is the true path always a minimum of S, or only a stationary point?
   - Solve a few problems twice, once with forces and once with a Lagrangian: the double Atwood machine, a block on a wedge that is free to slide, a bead on a rotating hoop, a cylinder rolling down an incline. Which forces never appear in the Lagrangian, and why?

3. **Symmetries and conservation laws.** *Find where conservation laws come from.*

   - Show that a coordinate missing from L gives a conserved momentum, and that a Lagrangian with no explicit dependence on time conserves the energy.
   - State and prove Noether's theorem in its simplest form. Which symmetry gives the conservation of angular momentum? Recover the effective potential of the Kepler problem.

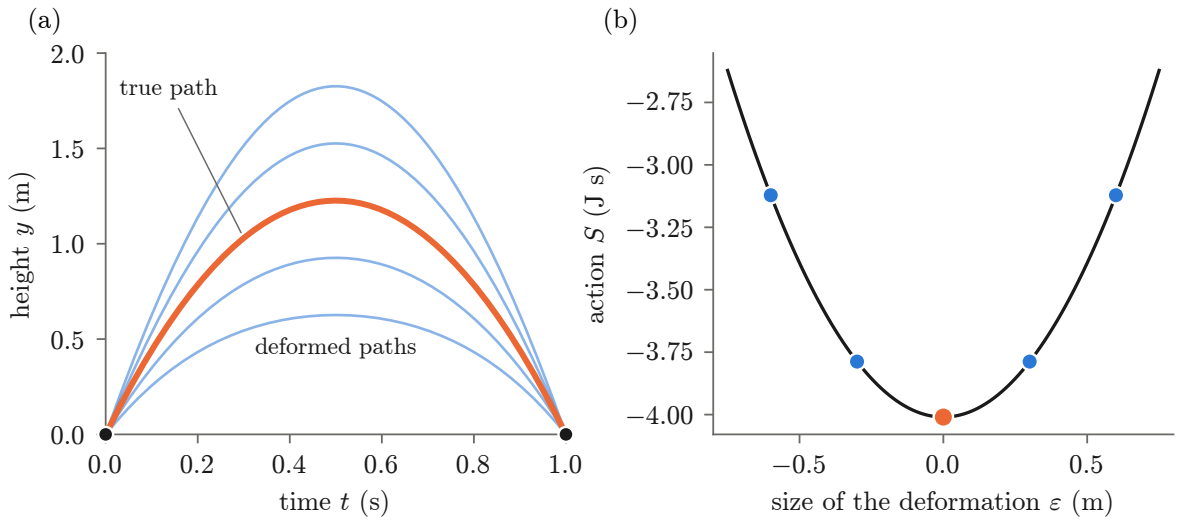


Figure 2. A ball of mass 1 kg thrown straight up and caught one second later. (a) The true path (orange) and four deformed paths (blue) with the same endpoints. (b) The action of each path as a function of the size of the deformation.

4. **The double pendulum.** *Derive the equations of motion and study small oscillations.*

   - Two masses m_1, m_2 hang on arms of length l_1, l_2 at angles θ_1, θ_2 from the vertical. Write T and V, and derive the equations of motion.
   - For small angles the equations become linear. Find the normal modes and their frequencies as an eigenvalue problem. What does each mode look like?

5. **Simulation.** *Explore large swings, where no formula exists.*

   - Write a fourth-order Runge–Kutta integrator from scratch (about twenty lines of Python, no library solvers) and test it on the conservation of energy.
   - Release two pendulums from almost the same position and measure how fast they move apart (Figure 1).
   - Map which releases lead to a flip, and after how long (Figure 3). Why does a whole region never flip?

6. **Further directions.**

   - Hamiltonian mechanics: positions and momenta on the same footing, and motion as a flow in phase space. Why do some numerical methods keep the energy almost constant for millions of steps while others slowly drift?
   - Learn the Lagrangian from data: train a neural network on observed trajectories, with the double pendulum as a test case.

Why Lagrangians?

Every fundamental theory of physics is written today as a principle of least action: electromagnetism, general relativity, and quantum mechanics in the form given to it by Feynman, where a particle explores all possible paths at once. Anyone who goes on with physics meets Lagrangians in every course.

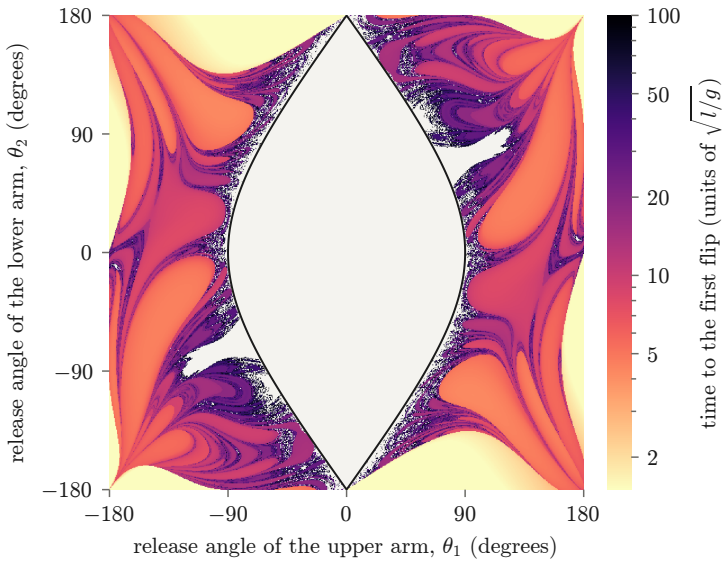


Figure 3. Double pendulums with equal masses and arms, released from rest at angles θ_1, θ_2 from the downward vertical. The colour shows the time before one of the arms first swings over the top. Pale grey means no flip within 100 time units. The black curve is $2\cos\theta_1 + \cos\theta_2 = 1$.

The mathematics also travels well. The Euler–Lagrange equation is the basic tool of dynamic optimisation, and in economics it gives the Euler equation that describes how consumption is spread over time. A double pendulum is the simplest model of a robot arm with two joints, and the equations of motion of robots are derived in exactly this way. Chaos, finally, is not a curiosity of pendulums: the same sensitivity to initial conditions is what limits weather forecasts to a couple of weeks.

Expected outcome.

A short illustrated report with:

- the Euler–Lagrange equation and Noether's theorem, derived from the principle of least action;
- a collection of problems solved both with forces and with Lagrangians;
- a study of the double pendulum: normal modes, sensitivity to initial conditions and a map of its flips, from a tested simulation.

Theory, problem collection and simulation are separate strands, so the work divides naturally among participants.

Prerequisites / Difficulty.

- Mechanics: kinetic and potential energy, Newton's laws, the simple pendulum.
- Analysis: partial derivatives, the chain rule, integration by parts.
- Linear algebra: eigenvalues of a 2×2 matrix.
- Basic Python and NumPy. No experience with numerical methods is needed.

Intermediate. No previous knowledge of analytical mechanics is needed. Steps 1–5 need only Physics I and the first analysis courses. The further directions are more demanding.

Starting materials.

Main reference. David Morin (2008). *Introduction to Classical Mechanics*, Chapter 6, The Lagrangian method, available as a sample chapter on the author's page. It starts from the Euler–Lagrange equations

(§6.1) and the principle of stationary action (§6.2), with many solved problems.

Lecture notes. David Tong, Lectures on Classical Dynamics, Chapter 2, free online: the principle of least action, constraints, Noether's theorem, and the double pendulum as an example of small oscillations (§2.6.1).

A first read. R. Feynman, *The Feynman Lectures on Physics*, Vol. II, Chapter 19, The Principle of Least Action, free online. A single lecture, almost without prerequisites, and the best place to start.

The double pendulum. T. Shinbrot, C. Grebogi, J. Wisdom and J. Yorke (1992). Chaos in a double pendulum. *American Journal of Physics* 60, 491–499. R. Levien and S. Tan (1993). Double pendulum: An experiment in chaos. *American Journal of Physics* 61, 1038–1044.

Going further. M. Cranmer et al. (2020). Lagrangian Neural Networks, for the last of the further directions.

Contacts.

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